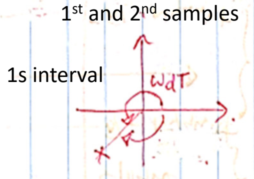
EN5101 Digital Control Systems
Aliasing and Frequency Warping

Prof. Rohan Munasinghe
Dept of Electronic and Telecommunication Engineering
University of Moratuwa

Aliasing

for $h(s)$, damped frequency $\omega_d = 4$ (rad/s) $= \frac{2}{\pi} = 0.6$Hz
under sampling at 1Hz (less than twice of ω_d)
discrete-time pole angle $= \omega_d T$ Then,



$\omega_a T = 2\pi - \omega_d T$

$\omega_a = \frac{2\pi}{T} - \omega_d$

$\omega_a = \omega_s - \omega_d$

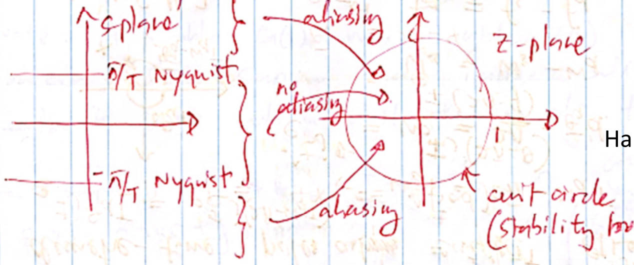
or $f_a = f_s - f_d$ — damped frequency; f_s — sampling frequency; f_a — aliasing frequency

Eg: $\omega_d = 4$ [rad/s]
$\omega_s = 2\pi$ [=1 Hz]
$\therefore \omega_a = \omega_s - \omega_d$
$f_a = f_s - f_d$
$= 1 - 4/2\pi$ = 0.27 Hz (-ve frequency)

0.6Hz signal if samples with 1Hz sampler generates a -0.27Hz signal (an alias)

Aliasing

Note: Sampling at rate of f_s maps all the frequencies into the unit circle if $h(s)$ is stable. Therefore it is required to check whether sampling frequency is adequate so that aliasing cannot happen.

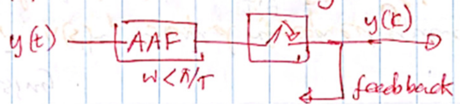


$\omega_s = 2\pi f_s$

$\omega_s = \frac{2\pi}{T_s}$

Half sampling rate $= \frac{\pi}{T_s}$

Solution: use anti-aliasing filter before sampler

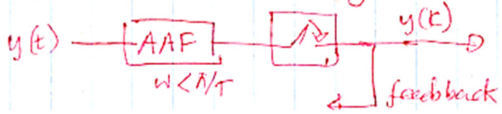


How to avoid Aliasing

Method 1 : High speed sampling (twice as fast as the highest signal frequency)

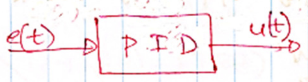
Method 2 : Anti-aliasing filter

Solution: use anti-aliasing filter before sampler



Digital Approximations and Stability

- Design $C(s)$, and approximate with $C(z)$.



$u(t) = K_p e(t) + K_d \dot{e}(t) + k_I \int e t\, dt$

PID Controller.

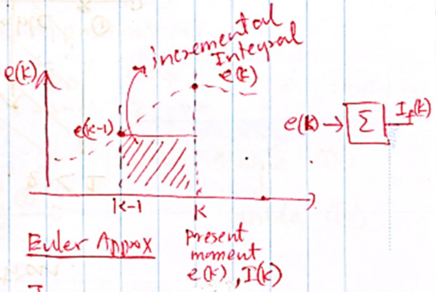
$U(s) = K_p E(s) + s K_D E(s) + \frac{1}{s} K_I E(s)$

Implement $s, \frac{1}{s}$ using approximate difference eq^{ns}

derivative

Integration.

Integration in DT. (FD)



* Forward difference Approx.

$I_f(k) = I_f(k-1) + e(k-1)T$

$I_f(z) = z^{-1} I_f(z) + z^{-1} E(z) T$

$(z-1) I_f(z) = T E(z)$

$I_f(z) = \frac{T}{z-1} E(z) \quad \therefore \frac{1}{s} \simeq \frac{T}{z-1}$

Euler Approx

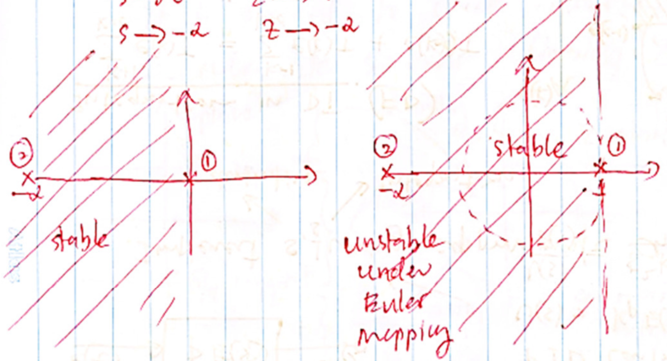
Discrete Integration of Forward Difference

$s \simeq \frac{z-1}{T} \Rightarrow z = 1 + Ts$

$s \to 0 \quad z \to 1$

$s \to -\infty \quad z \to -\infty$

$s < 0 \Rightarrow z < 1$



Note: A stable G(s) can be destabilized by the Euler Approximation of integration.

Integration in DT (BD)

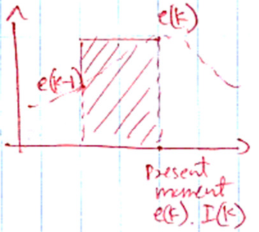
$I_b(k) = I_b(k-1) + e(k)T$

$I_b(z) = z^{-1} I_b(z) + E(z)T$

$(z-1) I_b(z) = zT E(z)$

$I_b(z) = \frac{zT}{z-1} E(z)$

$\therefore \frac{1}{s} \simeq \frac{zT}{z-1}$

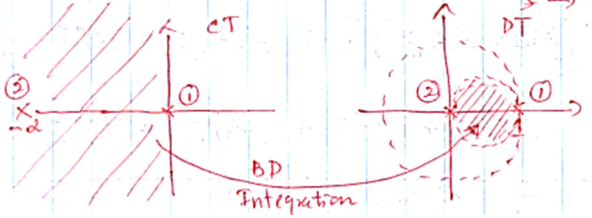


Stability of Backward difference Approximation.

$s \simeq \frac{z-1}{Tz} \Rightarrow z \simeq \frac{1}{1-Ts}$

$s \to 0 \quad z \to 1$

$s \to -\infty \quad z \to 0$



Note: stable C(s)

↓

stable C(z)

Forward and Backward Difference in Digital approximation

- Forward difference $\quad e(k) = \dfrac{I(k+1) - I(k)}{T}$

OR

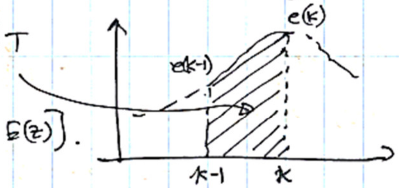
$$e(k-1) = \frac{I(k) - I(k-1)}{T} \quad \Rightarrow \quad \boldsymbol{I(k) = I(k-1) + e(k-1)T}$$

- Backward difference method $\quad e(k) = \dfrac{I(k) - I(k-1)}{T}$

$$\boldsymbol{I(k) = I(k-1) + e(k)T}$$

Integration in DT (Trapezoidal Approximation)

$$I_t(k) = I_t(k-1) + \frac{[e(k-1)+e(k)]}{2}T$$



Z

$$I_t(z) = z^{-1}I_t(z) + \frac{T}{2}\left[z^{-1}E(z) + E(z)\right].$$

$$(z-1)\,I(z) = \frac{T}{2}.\ (z+1)\,E(z)$$

$$I(z) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)E(z)$$

* Bilinear Approx.
* Tustin "

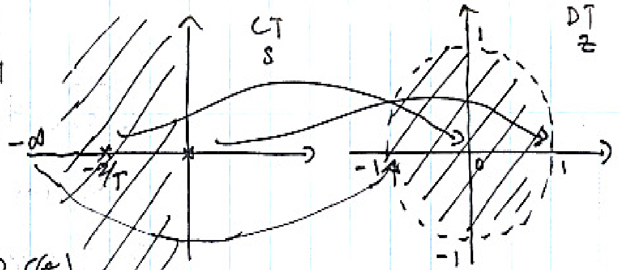
$$\therefore \quad \frac{1}{s} \simeq \left(\frac{z+1}{z-1}\right)\frac{T}{2}$$

stability of Trapezoidal Approximation.

$$s \simeq \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$\Rightarrow z \simeq \frac{2+Ts}{2-Ts}$$

$s \rightarrow 0 \quad z \rightarrow 1$
$s \rightarrow -\infty \quad z \rightarrow -1$
$s \rightarrow -2/T \quad z \rightarrow 0$



if $C(s)$ is stable $\Rightarrow$ $C(z)$ is stable

note: Tustin Approximation is the most commonly used integration Technique in DT.

Frequency Distortion in Digital Approximation Tustin

$z = e^{sT} \qquad s = j\omega$ in CT

Tustin $\rightarrow$ $z = e^{j\Omega T}$ $\rightarrow$ Ω resulting DT frequency of the mapping

$$\therefore \text{ as } \quad s = \frac{2(z-1)}{T(z+1)}$$

$$j\omega = \frac{2}{T}\cdot\frac{e^{j\Omega T}-1}{e^{j\Omega T}+1}$$

$\times\, e^{-j\Omega T/2}$

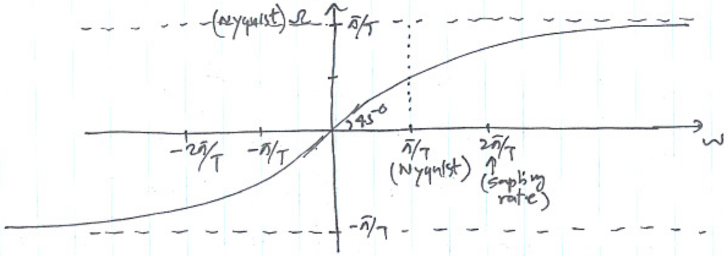
$$j\omega = \frac{2}{T}\cdot\frac{e^{j\Omega T/2} - e^{-j\Omega T/2}}{e^{j\Omega T/2} + e^{-j\Omega T/2}}$$

$$j\omega = \frac{2}{T}\cdot\frac{2j\sin(\Omega T/2)}{2\cos(\Omega T/2)}$$

$$\omega = \frac{2}{T}\cdot\tan(\Omega T/2)$$

$$\Omega = \frac{2}{T}\tan^{-1}\left(\frac{\omega T}{2}\right)$$

$\omega \rightarrow 0 \quad \Omega \rightarrow 0$ (no distortion)
$\omega \rightarrow \infty \quad \Omega \rightarrow \pi/T$ (= $\omega_s/2$ half sampling rate)



note: use Tustin approximation in DT integration to avoid stability issues, while maintaining better accuracy (less distortion)

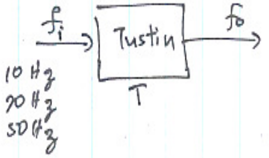
All the techniques in DT integration have distortions at higher frequencies. Specially around and above Nyquist frequency. Therefore, it is recommended to use substantially higher Sampling freq.

Class Exercise

Calculate frequency distortion at 10 Hz, 20 Hz and 50 Hz when sampling takes place at 120 Hz and 250 Hz, and 1K Hz

$$\Omega = \frac{2}{T}\tan^{-1}\left(\frac{\omega T}{2}\right)$$

$$f_o = \frac{f_s}{\pi}\tan^{-1}\left(\frac{\pi f_i}{f_s}\right)$$



A. Sampling at 120 Hz (just above twice the max frequency 50 Hz)

$$f_{10} = \frac{120}{\pi}\tan^{-1}\left(\frac{\pi 10}{120}\right) = 9.78 \text{ Hz}$$

$$f_{20} = \frac{120}{\pi}\tan^{-1}\left(\frac{\pi 20}{120}\right) = 18.42 \text{ Hz}$$

$$f_{50} = \frac{120}{\pi}\tan^{-1}\left(\frac{\pi 120}{50}\right) = 54.96 \text{ Hz}$$

} some distortion is there, may not be acceptable

B. Sampling at 250 Hz (sufficiently above twice the max frequency 50 Hz)

$$f_{10} = \frac{250}{\pi}\tan^{-1}\left(\frac{\pi . 10}{250}\right) = 9.95 \text{ Hz}$$

$$f_{20} = \frac{250}{\pi}\tan^{-1}\left(\frac{\pi . 20}{250}\right) = 19.59 \text{ Hz}$$

$$f_{50} = \frac{250}{\pi}\tan^{-1}\left(\frac{\pi . 50}{250}\right) = 44.64 \text{ Hz}$$

} → some distortion can be acceptable

C. Sampling at 1000 Hz (

$$f_{10} = \frac{1000}{\pi}\tan^{-1}\left(\frac{\pi . 10}{1000}\right) = 10. \text{ Hz}$$

$$f_{20} = \frac{1000}{\pi}\tan^{-1}\left(\frac{\pi . 20}{1000}\right) = 19.97 \text{ Hz}$$

$$f_{50} = \frac{1000}{\pi}\tan^{-1}\left(\frac{\pi . 50}{1000}\right) = 49.59 \text{ Hz}$$

} no distortion

Note: If we prefer to use a lower sampling rate, still reduce frequency distortion (at a given frequency) we could do what is known as "Prewarping"

Prewarping

Tustin

$$I_t(k) = I_t(k-1) + \frac{[e(k-1)+e(k)]T}{2}$$

lets use DT integration (Bilinear)

$$I_t^P(k) = I_t^P(k-1) + \frac{1}{\alpha}[e(k)+e(k-1)]$$

Then

$$I_t^P(z) = \frac{1}{\alpha}\left(\frac{z+1}{z-1}\right)E(z)$$

Tustin

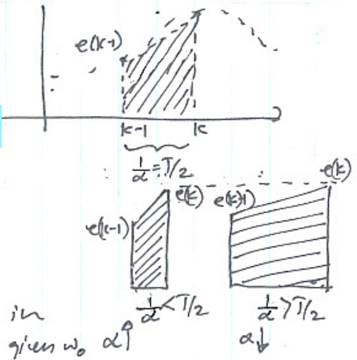
$$I(z) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)E(z)$$

$$\therefore \quad \frac{1}{s} \simeq \frac{1}{\alpha}\left(\frac{z+1}{z-1}\right)$$

Tustin

$$\frac{1}{s} \simeq \left(\frac{z+1}{z-1}\right)\frac{T}{2}$$

$$s \simeq \alpha\left(\frac{z-1}{z+1}\right)$$



Now we can select α so that in frequency mapping $\Omega^P = \omega$ at any given ω_0 α↑ α↓

$$\omega = \alpha \tan\left(\frac{\Omega^P T}{2}\right)$$ is the mapping and when $\omega = \omega_0$

$$\omega_0 = \alpha \tan\left(\frac{\omega_0 T}{2}\right)$$ ω_0 can be phase margin frequency, frequency of a critical notch, or an important oscillatory frequency.

$$\alpha_{\omega_0} = \frac{\omega_0}{\tan(\omega_0 T/2)} \Rightarrow \alpha_{f_0} = \frac{2\pi f_0}{\tan(\pi f_0/f_s)}$$

Class Exercise

eg Calculate α for no distortion at 50 Hz when sampling takes place at 250 Hz. Calculate the distortion at 10 Hz.

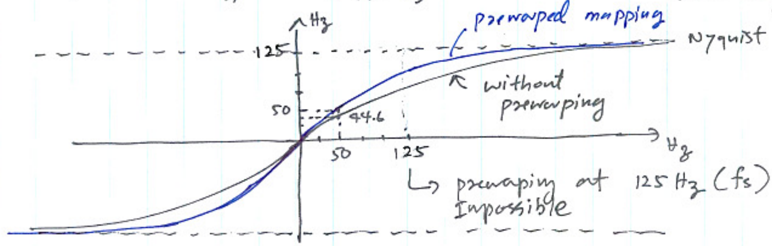
$$\alpha_{50Hz} = \frac{2\pi 50}{\tan\left(\frac{\pi 50}{250}\right)} = 432.4$$

distortion at 10 Hz

$$\Omega^P = \frac{2}{T}\tan^{-1}(\omega/\alpha)$$

$$f^P = \frac{f_s}{\pi}\tan^{-1}\left(\frac{2\pi f}{\alpha}\right)$$

$f = 10$ Hz $\quad f^P = \frac{250}{\pi}\tan^{-1}\left(\frac{2\pi 10}{432.4}\right) = 11.48$ Hz (frequency is lifted)



↳ prewarping at 125 Hz (fs) is Impossible

Take-home Assignment

- Draw frequency distortion graph of Tustin approximation for sampling at 250Hz
- Draw frequency distortion of Tustin approximation for sampling at 250Hz and prewarped at 10Hz.
- Compare and comment on the two frequency distortion graphs